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2010 J. Phys. A: Math. Theor. 43 095302

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# An anhomomorphic logic for quantum mechanics

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Received 28 October 2009, in final form 18 January 2010

Published 9 February 2010

Online at [stacks.iop.org/JPhysA/43/095302](http://stacks.iop.org/JPhysA/43/095302)

## Abstract

Although various schemes for anhomomorphic logics for quantum mechanics have been considered in the past we shall mainly concentrate on the quadratic or grade-2 scheme. In this scheme, the grade-2 truth functions are called coevents. We discuss properties of coevents, projections on the space of coevents and the master observable. We show that the set of projections forms an orthomodular poset. We introduce the concept of precluding coevents and show that this is stronger than the previously studied concept of preclusive coevents. Precluding coevents are defined naturally in terms of the master observable. A result that exhibits a duality between preclusive and precluding coevents is given. Some simple examples are presented.

PACS numbers: 02.10.Ab, 10.10.De, 02.10.Ud, 03.65.Ta

## 1. Introduction

The study of anhomomorphic logics for quantum mechanics was initiated by Sorkin [13]. Since then, other investigations in the subject have been carried out [5, 8, 12, 14–16]. This work has usually been conducted in relation to the subject of quantum measure theory and is mainly motivated by the histories approach to quantum mechanics and quantum gravity [4, 6, 14]. One of the objectives of this subject is to describe the possible physical realities and to identify the actual physical reality.

The basic structure is given by a set of outcomes  $\Omega$  together with an algebra  $\mathcal{A}$  of subsets of  $\Omega$  whose elements are called events. It is generally agreed that possible realities are described by 1–0 functions from  $\mathcal{A}$  to  $\mathbb{Z}_2$  called truth functions. There are various schemes for choosing truth functions that correspond to possible realities [5, 13]; two of the most popular have been the linear and multiplicative schemes [5, 14, 15]. We shall mainly concentrate on the quadratic scheme which has been rejected in the past but which we believe should be reconsidered. The elements of the chosen scheme are called coevents. Various methods have been devised for filtering out the unwanted coevents and selecting the actual reality. Three of these are called unitality, minimality and preclusivity [5, 14–16].

The reader certainly has the right to ask what should one expect from an anhomomorphic logic  $\mathcal{A}^*$ ? The answer is that  $\mathcal{A}^*$  should provide the tools for finding the actual reality  $\phi_a$  for a physical system  $(\Omega, \mathcal{A})$ . The reality  $\phi_a \in \mathcal{A}^*$  can be thought of as a truth function  $\phi_a : \mathcal{A} \rightarrow \{0, 1\}$  in the sense that  $\phi_a(A) = 1$  for an event  $A \in \mathcal{A}$  if and only if  $A$  actually happens ( $A$  occurs). The anhomomorphic logic  $\mathcal{A}^*$  restricts the possible realities from which  $\phi_a$  is selected to those that have desirable properties based on physical principles. The dynamics of the system is described by a quantum measure  $\mu$  on  $\mathcal{A}$  where  $\mu(A)$  is the ‘propensity’ that  $A \in \mathcal{A}$  occurs [13, 14]. We are reluctant to call  $\mu$  a quantum probability because  $\mu$  does not have the additivity property usually associated with a probability. An old principle of Carnot states that very unlikely events do not occur [5, 13]. We say that a truth function  $\phi$  is preclusive if  $\mu(A) = 0$  implies that  $\phi(A) = 0$  and postulate that  $\phi_a$  is preclusive [13, 14]. In general, there are many preclusive potential realities. For this reason, we restrict our search for  $\phi_a$  by considering the ‘correct’ anhomomorphic logic.

In section 2 we discuss the various schemes and give a reason for choosing the quadratic scheme. We call quadratic truth functions (grade-2) coevents. Section 3 discusses properties of these coevents. In section 4, we consider projections on the space  $\mathcal{A}^*$  of coevents and observables. We show that the set of projections forms an orthomodular poset. We introduce the concept of the master observable and present its properties. Section 5 considers the concept of preclusivity. Preclusive coevents have already been discussed in the literature, and we introduce a stronger notion that we call precluding coevents. This notion is defined naturally in terms of the master observable. We close this section with some simple examples and a result that exhibits a duality between preclusive and precluding coevents. For simplicity, the outcomes space  $\Omega$  will be assumed to have finite cardinality.

## 2. Truth functions

Let  $\Omega = \{\omega_1, \dots, \omega_n\}$  be the sample space for some physical experiment or situation. We call the elements of  $\Omega$  *outcomes* and for simplicity we take  $\Omega$  to be finite. The outcomes could correspond to particle locations or spin outcomes or particle trajectories, or fine-grained histories, etc. Subsets of  $\Omega$  are called *events* and we denote the set of all events  $2^\Omega$  by  $\mathcal{A}$ . We use the notation  $AB$  for the intersection  $A \cap B$  and if  $A \cap B = \emptyset$  we write  $A \cup B = A \cup B$ . We also use the notation  $A + B$  for the symmetric difference  $(AB') \cup (A'B)$  where  $A'$  denotes the complement of  $A$ .

The logic for  $\mathcal{A}$  gives the contact with reality; that is, the logic describes what actually happens. What actually happens may be determined by a truth function or 1–0 function  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$ . If  $\phi(A) = 1$ , then  $A$  happens or  $A$  is true and if  $\phi(A) = 0$ , then  $A$  does not happen or  $A$  is false. Other terminology that is used is that  $A$  occurs or does not occur. Now there are various admissible 1–0 functions depending on the situation or state of the system. For classical logic it is assumed that  $\phi$  is a homomorphism. That is:

- (1)  $\phi(\Omega) = 1$  (*unital*)
- (2)  $\phi(A + B) = \phi(A) + \phi(B)$  (*additive*)
- (3)  $\phi(AB) = \phi(A)\phi(B)$  (*multiplicative*).

Of course, in  $\mathbb{Z}_2 = \{0, 1\}$  addition is modulo 2. If  $\phi$  is a homomorphism, it can be shown that there exists an  $\alpha \in \Omega$  such that  $\phi(A) = 1$  if and only if  $\alpha \in A$ . Defining the *containment map*  $\alpha^* : \mathcal{A} \rightarrow \mathbb{Z}_2$  by

$$\alpha^*(A) = \begin{cases} 1 & \text{if } \alpha \in A \\ 0 & \text{if } \alpha \notin A, \end{cases}$$

we have that  $\phi = \alpha^*$ . This is eminently reasonable for classical mechanics. For example, a classical particle is definitely at a specific location  $\alpha \in \Omega$  at any given time.

However, in quantum mechanics, assuming that  $\phi$  must be a homomorphism can result in a contradiction. For example, consider a three-slit experiment where  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  and  $\omega_i$  is the outcome that a quantum particle impinges the detection screen at a fixed small region  $\Delta$  after going through slit  $i, i = 1, 2, 3$  [14, 15]. Then it is possible for

$$\phi(\{\omega_1, \omega_2\}) = \phi(\{\omega_2, \omega_3\}) = 0.$$

If  $\phi$  were a homomorphism, it follows that  $\phi = 0$ , i.e.  $\phi(A) = 0$  for all  $A \in \mathcal{A}$ . Thus, nothing happens. Mathematically this gives a contradiction because by (1),  $\phi(\Omega) = 1$ . This also gives a physical contradiction because there are certainly circumstances in which the particle is observed in  $\Delta$ . We again have that  $\phi(\Omega) = 1$ .

The fundamental question becomes: What are the admissible 1–0 functions for quantum mechanics? We have seen that there are  $n$  different homomorphisms corresponding to the  $n$  classical states and we have argued that (1), (2) and (3) are too restrictive for quantum mechanics. On the other hand, there are  $2^{2^n}$  possible 1–0 functions on  $\mathcal{A}$  and if we allow all of them, then the logic will have nothing to say. Thus, to have viable theory some restrictions must be put into place. In past studies, (1) is usually retained and either (2) or (3) are assumed [5, 14, 15]. In this work we shall not assume (1), (2) or (3) but shall postulate a generalization of (2). We shall also give an argument for the plausibility of this postulate. Since it would be unreasonable to consider a 1–0 function  $\phi$  that satisfies  $\phi(\emptyset) = 1$  whenever we write  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$ , we are assuming that  $\phi(\emptyset) = 0$ .

But first it is instructive to examine the form of 1–0 functions that satisfy (2) or (3). If  $\phi, \psi : \mathcal{A} \rightarrow \mathbb{Z}_2$ , we define their *sum* and *product* by  $(\phi + \psi)(A) = \phi(A) + \psi(A)$  and  $\phi\psi(A) = \phi(A)\psi(A)$ . Of course,  $\phi + \psi$  and  $\phi\psi$  are again 1–0 functions. We can form polynomials in the containment maps  $\alpha^*$  for  $\alpha \in \Omega$ . For example,

$$\alpha^* + \beta^* + \alpha^*\beta^* + \alpha^*\gamma^* + \alpha^*\beta^*\gamma^*$$

is a degree-3 polynomial. Since there are  $2^{2^n}$  different polynomials, we conclude that every  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  can be uniquely represented by a polynomial (up to an ordering of the terms). The proofs of parts of the following theorem are contained in [5, 14, 15]. Also, this theorem and theorems 3.1, 3.3 and 3.6 are special cases of more general results in the field of combinatorial polarization ([2] and references therein). We include the proofs for the reader's convenience because they are shorter and more direct than the proofs for the more general results.

**Theorem 2.1.** (a) A nonzero  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  satisfies (2) if and only if  $\phi = \alpha_1^* + \dots + \alpha_m^*$  for some  $\alpha_1, \dots, \alpha_m \in \Omega$ . (b)  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  with  $\phi \neq 0, 1$  satisfies (3) if and only if  $\phi = \alpha_1^* \dots \alpha_m^*$  for some  $\alpha_1, \dots, \alpha_m \in \Omega$ .

**Proof.** (a) We first show that  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is additive if and only if  $\phi$  satisfies

$$(4) \phi(A \cup B) = \phi(A) + \phi(B) \text{ for all disjoint } A, B \in \mathcal{A}.$$

If  $\phi$  is additive, then clearly  $\phi$  satisfies (4). Conversely, if  $\phi$  satisfies (4) then for any  $A, B \in \mathcal{A}$  we have

$$\phi(A) = \phi(AB' \cup AB) = \phi(AB') + \phi(AB).$$

Therefore,

$$\begin{aligned} \phi(A + B) &= \phi(AB' \cup A'B) = \phi(AB') + \phi(A'B) \\ &= \phi(AB') + \phi(AB) + \phi(AB) + \phi(A'B) \\ &= \phi(A) + \phi(B), \end{aligned}$$

so  $\phi$  is additive. Now suppose  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is additive and nonzero. Then there exist  $\alpha_1, \dots, \alpha_m \in \Omega$  such that  $\phi(\alpha_i) = 1, i = 1, \dots, m$  and  $\phi(\omega) = 0$  for  $\omega \in \{\alpha_1, \dots, \alpha_m\}'$  where for simplicity we write  $\phi(\omega) = \phi(\{\omega\})$ . By (4), for an  $A \in \mathcal{A}$  we have

$$\phi(A) = \sum_{\omega_i \in A} \phi(\omega_i) = \sum_{\alpha_i \in A} \phi(\alpha_i) = \sum_{\alpha_i \in A} 1 = \sum_{i=1}^m \alpha_i^*(A).$$

Hence,  $\phi = \alpha_1^* + \dots + \alpha_m^*$  and the converse is clear.

(b) If  $\phi = \alpha_1^* \dots \alpha_m^*$ , then

$$\begin{aligned} \phi(AB) &= \alpha_1^*(AB) \dots \alpha_m^*(AB) = \alpha_1^*(A)\alpha_1^*(B) \dots \alpha_m^*(A)\alpha_m^*(B) \\ &= \alpha_1^*(A) \dots \alpha_m^*(A)\alpha_1^*(B) \dots \alpha_m^*(B) = \phi(A)\phi(B), \end{aligned}$$

so  $\phi$  is multiplicative. Conversely, suppose that  $\phi$  is multiplicative and  $\phi \neq 0, 1$ . If  $A \subseteq B$ , we have  $AB = A$  so that

$$\phi(A) = \phi(AB) = \phi(A)\phi(B) \leq \phi(B).$$

Since  $\phi \neq 0$ , there exists an  $A \in \mathcal{A}$  with  $\phi(A) = 1$ . Let

$$B = \cap \{A \in \mathcal{A} : \phi(A) = 1\}.$$

Then  $B$  is the smallest set with  $\phi(B) = 1$ ; that is,  $\phi(A) = 1$  if and only if  $B \subseteq A$ . Since  $\phi \neq 1, B \neq \emptyset$ . Letting  $B = \{\alpha_1, \dots, \alpha_m\}$ , we have that  $\phi(A) = 1$  if and only if  $\alpha_i \in A, i = 1, \dots, m$ . Hence,

$$\phi(A) = \alpha_1^* \dots \alpha_m^*(A).$$

□

It follows from theorem 2.1 that  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is a homomorphism if and only if  $\phi = \alpha^*$  for some  $\alpha \in \Omega$ . We now consider a generalization of the additivity condition (2). For  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$ , we define the  $m$ -point interference  $I_\phi^m$  as the map from  $m$ -tuples of distinct elements of  $\Omega$  into  $\mathbb{Z}_2$  given by

$$I_\phi^m(\alpha_1, \dots, \alpha_m) = \phi(\{\alpha_1, \dots, \alpha_m\}) + \phi(\alpha_1) + \dots + \phi(\alpha_m),$$

where  $m \in \mathbb{N}$  with  $m \geq 2$ . Since it is clear that  $\phi$  is additive if and only if  $I_\phi^m = 0$  for all  $m$  with  $2 \leq m \leq n$ , we see that  $I_\phi^m$  gives a measure of the amount that  $\phi$  deviates from being additive. An analogous definition is used to describe interference for quantum measures [8]. Our basic postulate is that  $m$ -point interference is governed by two-point interferences, in the sense that

$$I_\phi^m(\alpha_1, \dots, \alpha_m) = \sum_{i < j=1}^m I_\phi^2(\alpha_i, \alpha_j). \tag{2.1}$$

We call (2.1) for all  $2 \leq m \leq n$  the *two-point interference condition*.

We say that  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is *grade-2 additive* if

$$\phi(A \cup B \cup C) = \phi(A \cup B) + \phi(A \cup C) + \phi(B \cup C) + \phi(A) + \phi(B) + \phi(C)$$

for all mutually disjoint  $A, B, C \in \mathcal{A}$ . We also call (2) *grade-1 additivity* and clearly grade-1 additivity implies grade-2 additivity, but we shall see that the converse does not hold. One can also define higher grade additivities but these shall not be considered here [12].

**Theorem 2.2.** *A function  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is grade-2 additive if and only if  $\phi$  satisfies the two-point interference condition.*

**Proof.** We shall show in corollary 3.2 that  $\phi$  is grade-2 additive if and only if

$$\phi(\{\alpha_1, \dots, \alpha_m\}) = \sum_{i < j=1}^m \phi(\{\alpha_i, \alpha_j\}) + \frac{1}{2}[1 - (-1)^m] \sum_{i=1}^m \phi(\alpha_i) \tag{2.2}$$

for all  $m \in \mathbb{N}$  with  $2 \leq m \leq n$ . But (2.2) is equivalent to

$$\begin{aligned} I_\phi^m(\alpha_1, \dots, \alpha_m) + \sum_{i=1}^m \phi(\alpha_i) \\ = \sum_{i < j=1}^m I_\phi^2(\alpha_i, \alpha_j) + (m - 1) \sum_{i=1}^m \phi(\alpha_j) + \frac{1}{2}[1 - (-1)^m] \sum_{i=1}^m \phi(\alpha_i). \end{aligned} \tag{2.3}$$

Moreover, (2.3) is equivalent to

$$\begin{aligned} I_\phi^m(\alpha_1, \dots, \alpha_m) &= \sum_{i < j=1}^m I_\phi^2(\alpha_i, \alpha_j) + \left[ m + \frac{1}{2}(1 - (-1)^m) \right] \sum_{i=1}^m \phi(\alpha_i) \\ &= \sum_{i < j=1}^m I_\phi^2(\alpha_i, \alpha_j), \end{aligned}$$

which is the two-point interference condition. □

### 3. Grade-2 additivity

The two-point interference condition is analogous to an interference condition that holds for quantum measures [8] and in our opinion this condition should hold for all (finite) quantum systems. It follows from theorem 2.2 that the set of possible realities for a quantum system is described by the set  $\mathcal{A}^*$  of grade-2 additive functions from  $\mathcal{A}$  to  $\mathbb{Z}_2$ . We call the elements of  $\mathcal{A}^*$  coevents. We first give the result that was needed in the proof of theorem 2.2.

**Theorem 3.1.** *A map  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is a coevent if and only if  $\phi$  satisfies*

$$\phi(A_1 \cup \dots \cup A_m) = \sum_{i < j=1}^m \phi(A_i \cup A_j) + \frac{1}{2}[1 - (-1)^m] \sum_{i=1}^m \phi(A_i) \tag{3.1}$$

for all  $m \in \mathbb{N}$  with  $2 \leq m \leq n$ .

**Proof.** If (3.1) holds, then  $\phi$  is clearly a coevent. Conversely, assume that  $\phi$  is a coevent. We now prove (3.1) by induction on  $m$ . The result holds for  $m = 2, 3$ . Suppose the result holds for  $m \geq 2$ , where  $m$  is odd. Then

$$\begin{aligned} \phi \left( \bigcup_{i=1}^{m+1} A_i \right) &= \phi [A_1 \cup \dots \cup (A_m \cup A_{m+1})] \\ &= \sum_{i < j=1}^{m-1} \phi(A_i \cup A_j) + \sum_{i=1}^{m-1} \phi[A_i \cup (A_m \cup A_{m+1})] \\ &\quad + \sum_{i=1}^{m-1} \phi(A_i) + \phi(A_m \cup A_{m+1}) \\ &= \sum_{i < j=1}^{m-1} \phi(A_i \cup A_j) + \sum_{i=1}^{m-1} \phi(A_i \cup A_m) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{m-1} \phi(A_i \cup A_{m+1}) + \phi(A_m \cup A_{m+1}) \\
 & = \sum_{i < j=1}^{m+1} \phi(A_i \cup A_j).
 \end{aligned}$$

Suppose the result holds for  $m \geq 2$  where  $m$  is even. Then

$$\begin{aligned}
 \phi\left(\bigcup_{i=1}^{m+1} A_i\right) & = \phi[A_1 \cup \dots \cup (A_m \cup A_{m+1})] \\
 & = \sum_{i < j=1}^{m-1} \phi(A_i \cup A_j) + \sum_{i=1}^{m-1} \phi[A_i \cup (A_m \cup A_{m+1})] \\
 & = \sum_{i < j=1}^{m-1} \phi(A_i \cup A_j) + \sum_{i=1}^{m-1} \phi(A_i \cup A_m) + \sum_{i=1}^{m-1} \phi(A_i \cup A_{m+1}) \\
 & \quad + \phi(A_m \cup A_{m+1}) + \sum_{i=1}^{m-1} \phi(A_i) + \phi(A_m) + \phi(A_{m+1}) \\
 & = \sum_{i < j=1}^{m+1} \phi(A_i \cup A_j) + \sum_{i=1}^{m+1} \phi(A_i).
 \end{aligned}$$

The result now follows by induction. □

**Corollary 3.2.** *A map  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is a coevent if and only if (2.2) holds for all  $2 \leq m \leq n$ .*

**Proof.** If  $\phi$  is a coevent, then  $\phi$  satisfies (2.2) by letting  $A_i = \{\alpha_i\}$  in (3.1). Conversely, suppose  $\phi$  satisfies (2.2) and let  $A, B, C \in \mathcal{A}$  be mutually disjoint with  $A = \{\alpha_1, \dots, \alpha_r\}$ ,  $B = \{\beta_1, \dots, \beta_s\}$ ,  $C = \{\gamma_1, \dots, \gamma_t\}$ . The special cases in which at least one of the sets  $A, B$  or  $C$  has cardinality less than two are easily treated, so we assume their cardinalities are at least two. Then by (2.2) we have that

$$\begin{aligned}
 & \phi(A \cup B) + \phi(A \cup C) + \phi(B \cup C) + \phi(A) + \phi(B) + \phi(C) \\
 & = \sum_{i=1}^r \sum_{j=1}^s \phi(\{\alpha_i, \beta_j\}) + \frac{1}{2}[1 - (-1)^{r+s}] \left[ \sum_{i=1}^r \phi(\alpha_i) + \sum_{i=1}^s \phi(\beta_i) \right] \\
 & \quad + \sum_{i=1}^r \sum_{j=1}^t \phi(\{\alpha_i, \gamma_j\}) + \frac{1}{2}[1 - (-1)^{r+t}] \left[ \sum_{i=1}^r \phi(\alpha_i) + \sum_{i=1}^t \phi(\gamma_i) \right] \\
 & \quad + \sum_{i=1}^s \sum_{j=1}^t \phi(\{\beta_i, \gamma_j\}) + \frac{1}{2}[1 - (-1)^{s+t}] \left[ \sum_{i=1}^s \phi(\beta_i) + \sum_{i=1}^t \phi(\gamma_i) \right] \\
 & \quad + \sum_{i < j=1}^r \phi(\{\alpha_i, \alpha_j\}) + \frac{1}{2}[1 - (-1)^r] \sum_{i=1}^r \phi(\alpha_i) + \sum_{i < j=1}^s \phi(\{\beta_i, \beta_j\}) \\
 & \quad + \frac{1}{2}[1 - (-1)^s] \sum_{i=1}^s \phi(\beta_i) + \sum_{i < j=1}^t \phi(\{\gamma_i, \gamma_j\}) + \frac{1}{2}[1 - (-1)^t] \sum_{i=1}^t \phi(\gamma_i) \\
 & = \sum_{i < j=1}^r \phi(\{\alpha_i, \alpha_j\}) + \sum_{i < j=1}^s \phi(\{\beta_i, \beta_j\}) + \sum_{i < j=1}^t \phi(\{\gamma_i, \gamma_j\})
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^r \sum_{j=1}^s \phi(\{\alpha_i, \beta_j\}) + \sum_{i=1}^r \sum_{j=1}^t \phi(\{\alpha_i, \gamma_j\}) + \sum_{i=1}^s \sum_{j=1}^t \phi(\{\beta_i, \gamma_j\}) \\
 & + \frac{1}{2} [1 - (-1)^{r+s+t}] \left[ \sum_{i=1}^r \phi(\alpha_i) + \sum_{i=1}^s \phi(\beta_i) + \sum_{i=1}^t \phi(\gamma_i) \right] \\
 & = \phi(A \cup B \cup C),
 \end{aligned}$$

where the second to last equality comes from the fact that in  $\mathbb{Z}_2$  we have

$$\begin{aligned}
 & \frac{1}{2} [1 - (-1)^{r+s}] (a + b) + \frac{1}{2} [1 - (-1)^{r+t}] (a + c) + \frac{1}{2} [1 - (-1)^{s+t}] (b + c) \\
 & + \frac{1}{2} [1 - (-1)^r] a + \frac{1}{2} [1 - (-1)^s] b + \frac{1}{2} [1 - (-1)^t] c \\
 & = \frac{1}{2} [1 - (-1)^{r+s+t}] (a + b + c)
 \end{aligned}$$

for all  $r, s, t \in \mathbb{N}, a, b, c \in \mathbb{Z}_2$  which can be checked by cases. □

We call the set of coevents  $\mathcal{A}^*$  an *anhomomorphic logic*. Various schemes for anhomomorphic logics have been developed in the literature [5, 14, 15]. In fact, the present scheme was rejected in [5] because in some examples there were not enough coevents available. The reason for this is that only minimal (or primitive) and unital coevents were considered. We disagree with this analysis and believe that these restrictions are completely unnecessary. However, we shall later consider another means for restricting coevents that has already been used, called preclusivity. We now give further properties of coevents.

**Theorem 3.3.** *A map  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is a coevert for  $\Omega = \{\omega_1, \dots, \omega_n\}$  if and only if  $\phi$  is a first or second degree polynomial in the  $\omega_i^*$ , that is*

$$\phi = \sum_{i=1}^n a_i \omega_i^* + \sum_{i,j=1}^n b_{ij} \omega_i^* \omega_j^*, \tag{3.2}$$

where  $a_i, b_{ij} \in \mathbb{Z}_2$ .

**Proof.** It is easy to check that  $\omega_i^* \omega_j^*$  are coevents and that the sum of coevents is a coevert. Hence, any map  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  of the form (3.2) is a coevert. Conversely, suppose  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is a coevert. Reorder the  $\omega_i$  if necessary so that

$$\begin{aligned}
 \phi(\omega_1) &= \dots = \phi(\omega_r) = 1, \phi(\{\omega_i, \omega_j\}) = \dots = \phi(\{\omega_{i'}, \omega_{j'}\}) = 1, i, j, i', j' \leq r \\
 \phi(\{\omega_s, \omega_t\}) &= \dots = \phi(\{\omega_{s'}, \omega_{t'}\}) = 1, s, s' \leq r, t, t' > r \\
 \phi(\{\omega_u, \omega_v\}) &= \dots = \phi(\{\omega_{u'}, \omega_{v'}\}) = 1, u, v, u', v' > r
 \end{aligned}$$

and  $\phi$  is 0 for all other singleton and doubleton sets. Define  $\psi : \mathcal{A} \rightarrow \mathbb{Z}_2$  by

$$\psi = \sum_{k=1}^r \omega_k^* + \omega_i^* \omega_j^* + \dots + \omega_{i'}^* \omega_{j'}^* + \omega_u^* \omega_v^* + \dots + \omega_{u'}^* \omega_{v'}^* + \sum_{k=1}^r \sum_{w \in W} \omega_k^* \omega_w^*,$$

where  $W$  is the set of indices that are not represented above. Then  $\phi$  and  $\psi$  are coevents that agree on singleton and doubleton sets. By (2.2)  $\phi$  and  $\psi$  coincide. □

We now illustrate theorem 3.3 with an example. Let  $\Omega = \{\omega_1, \dots, \omega_5\}$  and suppose  $\phi \in \mathcal{A}^*$  satisfies  $\phi(\omega_1) = \phi(\omega_2) = 1$ ,

$$\phi(\{\omega_1, \omega_2\}) = \phi(\{\omega_2, \omega_3\}) = \phi(\{\omega_4, \omega_5\}) = 1$$

and  $\phi$  is 0 for all other singleton and doubleton sets. Define  $\psi \in \mathcal{A}^*$  by

$$\psi = \omega_1^* + \omega_2^* + \omega_1^* \omega_2^* + \omega_4^* \omega_5^* + \omega_1^* \omega_3^* + \omega_1^* \omega_4^* + \omega_1^* \omega_5^* + \omega_2^* \omega_4^* + \omega_2^* \omega_5^*.$$



Then  $\phi$  and  $\psi$  are coevents that agree on singleton and doubleton sets so by (2.2)  $\phi$  and  $\psi$  coincide.

The next result follows from the proof of theorem 3.3.

**Corollary 3.4.** *Given any assignment of zeros and ones to the singleton and doubleton sets of  $\Omega = \{\omega_1, \dots, \omega_n\}$ , there exists a unique coevent  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  that has these values.*

It follows from theorem 3.3 that the anhomomorphic logic  $\mathcal{A}^*$  is a vector space over  $\mathbb{Z}_2$  with dimension

$$\dim(\mathcal{A}^*) = n + \binom{n}{2} = \frac{n(n+1)}{2}.$$

Hence, the cardinality of  $\mathcal{A}^*$  is  $2^{n(n+1)/2}$  which is much smaller than the cardinality  $2^{2^n}$  of the set of all 1-0 functions on  $\Omega$ .

**Lemma 3.5.** (a)  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is grade-1 additive if and only if  $\phi(A \cup B) = \phi(A) + \phi(B) + \phi(AB)$  for all  $A, B \in \mathcal{A}$  and  $\phi(\emptyset) = 0$ . (b)  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is grade-2 additive if and only if

$$\phi(A \cup B) = \phi(A) + \phi(B) + \phi(AB) + \phi(A+B) + \phi(AB') + \phi(A'B) \quad (3.8)$$

for all  $A, B \in \mathcal{A}$ .

**Proof.** (a) If  $\phi$  is grade-1 additive, then

$$\phi(A \cup B) = \phi(A+B) = \phi(A) + \phi(B).$$

Hence,

$$\phi(A) = \phi[(AB) \cup (AB')] = \phi(AB) + \phi(AB')$$

for all  $A, B \in \mathcal{A}$ . We conclude that

$$\begin{aligned} \phi(A \cup B) &= \phi(AB') + \phi(A'B) + \phi(AB) \\ &= \phi(A) + \phi(AB) + \phi(B) + \phi(AB) + \phi(AB) \\ &= \phi(A) + \phi(B) + \phi(AB). \end{aligned}$$

Also, it is clear that  $\phi(\emptyset) = 0$ . Conversely, suppose the given formulas hold. Then as before

$$\begin{aligned} \phi(A+B) &= \phi(AB') + \phi(A'B) = \phi(A) + \phi(AB) + \phi(B) + \phi(AB) \\ &= \phi(A) + \phi(B), \end{aligned}$$

so  $\phi$  is grade-1 additive. (b) If  $\phi$  is grade-2 additive, then

$$\begin{aligned} \phi(A \cup B) &= \phi[(AB) \cup (AB') \cup (A'B)] \\ &= \phi(A+B) + \phi(A) + \phi(B) + \phi(AB) + \phi(AB') + \phi(A'B). \end{aligned}$$

Conversely, if (3.8) holds, then letting  $A_1 = A \cup C, B_1 = B \cup C$  we have that

$$\begin{aligned} \phi(A \cup B \cup C) &= \phi(A_1 \cup B_1) \\ &= \phi(A_1 + B_1) + \phi(A_1) + \phi(B_1) + \phi(A_1 B'_1) + \phi(A'_1 B_1) + \phi(A_1 B_1) \\ &= \phi(A \cup B) + \phi(A \cup C) + \phi(B \cup C) + \phi(A) + \phi(B) + \phi(C), \end{aligned}$$

which is grade-2 additivity. □

In the next result,  $\mathcal{A} \times \mathcal{A}$  denotes the collection  $2^{\Omega \times \Omega}$  of all subsets of  $\Omega \times \Omega$ . It is easy to see that the map  $\lambda$  in this result is not unique.

**Theorem 3.6.**  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is a coevent if and only if there exists a grade-1 additive map  $\lambda : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{Z}_2$  such that  $\phi(A) = \lambda(A \times A)$  for all  $A \in \mathcal{A}$ .

**Proof.** If  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  is a coevent, then by theorem 3.3,  $\phi$  has the form

$$\phi = \sum \alpha_i^* + \sum \beta_i^* \gamma_i^*$$

for  $\alpha_i, \beta_i, \gamma_i \in \Omega$ . Define  $\lambda : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{Z}_2$  by

$$\lambda = \sum (\alpha_i \times \alpha_i)^* + \sum (\beta_i \times \gamma_i)^*.$$

Then by theorem 2.1(a),  $\lambda$  is grade-1 additive and by lemma 3.5(a),  $\lambda(A \times A) = \phi(A)$  for all  $A \in \mathcal{A}$ . Conversely, suppose  $\lambda : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{Z}_2$  is grade-1 additive. By theorem 2.1(a),  $\lambda$  has the form

$$\lambda = \sum (\alpha_i \times \beta_j)^*.$$

If  $\phi : \mathcal{A} \rightarrow \mathbb{Z}_2$  satisfies  $\phi(A) = \lambda(A \times A)$ , then

$$\begin{aligned} \phi(A) &= \sum (\alpha_i \times \beta_j)^*(A \times A) = \sum \alpha_i^*(A) \beta_j^*(A) \\ &= \sum \alpha_i^*(A) + \sum (\alpha_i^* \beta_j^*)(A), \end{aligned}$$

where the first summation on the right-hand side is when  $\alpha_i = \beta_j$ . It follows from theorem 2.1(b) that  $\phi$  is a coevent.  $\square$

We now briefly discuss the possible strange behavior of coevents. Taking the particle location interpretation, the ‘superposition’  $\omega_1^* + \omega_2^*$  states that the particle is at position 1 and at position 2 but not at 1 or 2. The ‘entanglement’  $\omega_1^* \omega_2^*$  states that the particle is at position 1 or at position 2 but if we look closely, it is not at either 1 or 2.

#### 4. Projections and observables

In the following,  $\Omega = \{\omega_1, \dots, \omega_n\}$  is a finite set,  $\mathcal{A}$  is the Boolean algebra of all subsets of  $\Omega$  and  $\mathcal{A}^*$  is the anhomomorphic logic. We have seen in section 3 that  $\mathcal{A}^*$  is an  $n(n+1)/2$ -dimensional vector space over  $\mathbb{Z}_2$  with basis consisting of the additive terms  $\omega_i^*$  and the quadratic terms  $\omega_i^* \omega_j^*$ . A *projection* on  $\mathcal{A}^*$  is a linear (or additive) idempotent map  $P : \mathcal{A}^* \rightarrow \mathcal{A}^*$ . That is,  $P(\phi + \psi) = P\phi + P\psi$  for all  $\phi, \psi \in \mathcal{A}^*$  and  $P^2 = PP = P$ . We denote the set of all projections on  $\mathcal{A}^*$  by  $\mathcal{P}(\mathcal{A}^*)$ . If  $P, Q \in \mathcal{P}(\mathcal{A}^*)$  with  $PQ = QP$ , then it is clear that  $P + Q$  and  $PQ$  are again projections. For  $P \in \mathcal{P}(\mathcal{A}^*)$ , we define  $P' \in \mathcal{P}(\mathcal{A}^*)$  by  $P' = I + P$ . For  $P, Q \in \mathcal{P}(\mathcal{A}^*)$  we define  $P \leq Q$  if  $PQ = QP = P$ . We call a partially ordered set a *poset*. The greatest lower bound and the least upper bound (if they exist) in a poset are denoted by  $P \wedge Q$  and  $P \vee Q$ , respectively. For related work, we refer the reader to [17]

**Theorem 4.1.** (a)  $(\mathcal{P}(\mathcal{A}^*), \leq)$  is a poset. (b) For  $P, Q \in \mathcal{P}(\mathcal{A}^*)$  we have that  $P'' = P$ ,  $P \wedge P' = 0$  and  $P \leq Q$  implies  $Q' \leq P'$ . (c) If  $PQ = QP$  then  $P \wedge Q = PQ$  and  $P \vee Q = P + Q + PQ$ .

**Proof.** (a) Clearly,  $P \leq P$  for all  $P \in \mathcal{P}(\mathcal{A}^*)$ . If  $P \leq Q$  and  $Q \leq P$ , then

$$P = PQ = QP = Q.$$

If  $P \leq Q$  and  $Q \leq R$  then

$$PR = PQR = PQ = P$$

and

$$RP = RQP = QP = P.$$

Hence,  $P \leq R$  so  $(\mathcal{P}(\mathcal{A}^*), \leq)$  is a poset. (b) Clearly  $P'' = P$ . If  $P \leq Q$ , then

$$(I + P)(I + Q) = I + P + Q + PQ = I + P + Q + P = I + Q.$$

Similarly,  $(I + Q)(I + P) = I + Q$  so  $Q' \leq P'$ . If  $Q \leq P$ ,  $P'$ , then

$$Q = QP' = Q(I + P) = Q + QP = Q + Q = 0.$$

Hence,  $P \wedge P' = 0$ . (c) Since

$$(PQ)P = (QP)P = QP = PQ,$$

we have that  $PQ \leq P$  and similarly  $PQ \leq Q$ . Suppose that  $R \in \mathcal{P}(\mathcal{A}^*)$  with  $R \leq P, Q$ . Then,

$$RPQ = RQ = R$$

so that  $R \leq PQ$ . Hence,  $P \wedge Q = PQ$ . By DeMorgan's law, we have that

$$\begin{aligned} P \vee Q &= (P' \wedge Q')' = (P'Q')' = I + (I + P)(I + Q) \\ &= I + I + P + Q + PQ = P + Q + PQ. \end{aligned}$$

□

A poset  $(\mathcal{P}, \leq)$  with a mapping  $' : \mathcal{P} \rightarrow \mathcal{P}$  satisfying the conditions of theorem 4.1(b) is called an *orthocomplemented poset*. If  $P \leq Q'$ , we write  $P \perp Q$  and say that  $P$  and  $Q$  are *orthogonal*. Of course,  $P \perp Q$  if and only if  $Q \perp P$ . An orthocomplemented poset  $(\mathcal{P}, \leq, ')$  is called an *orthomodular poset* if for  $P, Q \in \mathcal{P}$  we have that  $P \perp Q$  implies  $P \vee Q$  exists and  $P \leq Q$  implies

$$Q = P \vee (Q \wedge P').$$

**Theorem 4.2.** (a) For  $P, Q \in \mathcal{P}(\mathcal{A}^*)$ ,  $P \perp Q$  if and only if  $PQ = QP = 0$ . (b)  $(\mathcal{P}(\mathcal{A}^*), \leq, ')$  is an orthomodular poset.

**Proof.** (a) If  $P \perp Q$ , then

$$P = P(I + Q) = P + PQ.$$

Adding  $P$  to both sides gives  $PQ = 0$ . Similarly,  $QP = 0$ . If  $PQ = QP = 0$ , then

$$P(I + Q) = P + PQ = P.$$

Similarly,  $(I + Q)P = P$  so  $P \leq Q'$ . (b) If  $P \perp Q$ , then by (a) we have that  $PQ = QP = 0$ . Hence, by theorem 4.1(c) we conclude that  $P \vee Q$  exists and  $P \vee Q = P + Q$ . Now assume that  $P \leq Q$ . Since  $Q' \leq P'$ , we have that  $Q' \perp P$ . Hence, as before  $P \vee Q'$  exists. It follows that  $Q \wedge P' = (P \vee Q')'$  exists. Since

$$P \leq P \vee Q' = (Q \wedge P')',$$

we have that  $P \perp Q \wedge P'$ . Hence,  $P \vee (Q \wedge P') = P + Q \wedge P'$  exists. By theorem 4.1(c), we have that

$$Q \wedge P' = QP' = Q(I + P) = Q + PQ.$$

Therefore,

$$Q = P + (Q \wedge P') = P + Q + PQ = P + Q \wedge P' = P \vee (Q \wedge P').$$

It follows that  $(\mathcal{P}(\mathcal{A}^*), \leq, ')$  is an orthomodular poset. □

An orthomodular poset is frequently called a ‘quantum logic’. Quantum logics have been studied for over 45 years in the foundations of quantum mechanics [1, 3, 7, 9–11, 18]. It is interesting that the present formalism is related to this older approach. In the quantum logic approach, the elements of  $\mathcal{P}(\mathcal{A}^*)$  are thought of as quantum propositions or events. When we later consider observables, we shall see that there is a natural correspondence between elements of  $\mathcal{A}$  and some of the elements of  $\mathcal{P}(\mathcal{A}^*)$ . These elements of  $\mathcal{P}(\mathcal{A}^*)$  then become quantum generalizations of the events in  $\mathcal{A}$ . In accordance with the quantum logic approach we say that  $P, Q \in \mathcal{P}(\mathcal{A}^*)$  are *compatible* if there exist mutually orthogonal elements  $P_1, Q_1, R \in \mathcal{P}(\mathcal{A}^*)$  such that  $P = R_1 \vee R$  and  $Q = Q_1 \vee R$ . Compatible events describe events that can occur in a single measurement [9–11].

**Theorem 4.3.**  $P, Q \in \mathcal{P}(\mathcal{A}^*)$  are compatible if and only if  $PQ = QP$ .

**Proof.** If  $P, Q$  are compatible, there exist  $P_1, Q_1$  and  $R \in \mathcal{P}(\mathcal{A}^*)$  satisfying the given conditions. Then  $P = P_1 + R, Q = Q_1 + R$ , so by theorem 4.2(a) we have that

$$PQ = (P_1 + R)(Q_1 + R) = P_1Q_1 + P_1R + RQ_1 + R = R.$$

Similarly,  $QP = R$ . Conversely, suppose that  $PQ = QP$ . Define  $R = PQ, P_1 = P + PQ, Q_1 = Q + PQ$ . It is easy to check that  $P_1, Q_1$  and  $R$  are mutually orthogonal elements of  $\mathcal{P}(\mathcal{A}^*)$ . Applying theorem 4.1(c), we conclude that

$$P = (P + PQ) + PQ = P_1 + R = P_1 \vee R$$

and

$$Q = (Q + PQ) + PQ = Q_1 + R = Q_1 \vee R.$$

Hence,  $P$  and  $Q$  are compatible. □

In the quantum logic approach, instead of  $\mathcal{P}(\mathcal{A}^*)$  we frequently have the projective space  $\mathcal{P}(H)$  of orthogonal (self-adjoint) projections on a complex Hilbert space  $H$ . For  $P, Q \in \mathcal{P}(H)$  we define  $P \leq Q$  if  $P = PQ$ . It is well known that theorems 4.2 and 4.3 hold for  $(\mathcal{P}(H), \leq)$ . However, it does not immediately follow that these theorems hold for  $(\mathcal{P}(\mathcal{A}^*), \leq)$  because the structure of the vector space  $\mathcal{A}^*$  over  $\mathbb{Z}_2$  is quite different than that of an inner product space over the complex field  $\mathbb{C}$ . Also,  $\mathcal{P}(\mathcal{A}^*)$  consists of all projections on  $\mathcal{A}^*$  while  $\mathcal{P}(H)$  consists of only orthogonal projections. This is illustrated in example 1 at the end of this section.

For further emphasis, we give some examples of the differences between  $\mathcal{P}(\mathcal{A}^*)$  and  $\mathcal{P}(H)$ . For commuting projections  $P, Q \in \mathcal{P}(\mathcal{A}^*)$  we have  $P + Q \in \mathcal{P}(\mathcal{A}^*)$  which is not true in  $\mathcal{P}(H)$ . For  $P, Q \in \mathcal{P}(H)$  if  $PQ = 0$  then  $PQ = QP$  which is not true in  $\mathcal{P}(\mathcal{A}^*)$  as is shown in example 1 at the end of this section. Theorem 4.1(c) does not hold in  $(\mathcal{P}(H), \leq)$ . Finally, it is known that  $(\mathcal{P}(H), \leq)$  is a lattice ( $P \wedge Q$  and  $P \vee Q$  always exist). However, it is not known whether  $\mathcal{P}(\mathcal{A}^*)$  is a lattice and this would be an interesting problem to investigate.

We have seen that  $\{\omega_i^*, \omega_i^* \omega_j^* : i, j = 1, \dots, n\}$  forms a basis for the vector space  $\mathcal{A}^*$ . For  $\omega_i \in \Omega$  define the map  $P(\omega_i) : \mathcal{A}^* \rightarrow \mathcal{A}^*$  by  $P(\omega_i)\omega_j^* = \omega_i^* \omega_j^*$ ,

$$P(\omega_i)\omega_i^* \omega_j^* = P(\omega_i)\omega_j^* \omega_i^* = \omega_i^* \omega_j^*$$

and for  $i, j, k$  distinct  $P(\omega_i)\omega_j^* \omega_k^* = 0$  and extended  $P(\omega_i)$  to  $\mathcal{A}^*$  by linearity. It is easy to check that  $P(\omega_i) \in \mathcal{P}(\mathcal{A}^*), i = 1, \dots, n$ . Moreover, one can check that  $P(\omega_i) + P(\omega_j) \in \mathcal{P}(\mathcal{A}^*)$  and that  $P(\omega_i)P(\omega_j) = P(\omega_j)P(\omega_i) \in \mathcal{P}(\mathcal{A}^*)$ . For  $A \in \mathcal{A}$ , define  $P(A) : \mathcal{A}^* \rightarrow \mathcal{A}^*$  by

$$P(A) = \sum \{P(\omega_i) + P(\omega_i)P(\omega_j) : \omega_i, \omega_j \in A, i < j\}.$$

It follows that  $P(A) \in \mathcal{P}(\mathcal{A}^*)$  for all  $A \in \mathcal{A}$ . For example,

$$P(\{\omega_1, \omega_2, \omega_3\}) = P(\omega_1) + P(\omega_2) + P(\omega_3) + P(\omega_1)P(\omega_2) + P(\omega_1)P(\omega_3) + P(\omega_2)P(\omega_3).$$

The map  $P : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A}^*)$  given by  $A \mapsto P(A)$  is called the *master observable*. By convention,  $P(\emptyset) = 0$  and one can verify that  $P(A)P(B) = P(B)P(A)$  for all  $A, B \in \mathcal{A}$ .

In general,  $P(\cdot)$  is not additive or multiplicative. For example letting  $A = \{\omega_1\}, B = \{\omega_2\}$  we have that

$$P(AB) = 0 \neq \omega_1^* \omega_2^* = P(A)P(B)$$

and

$$\begin{aligned} P(A + B) &= P(\{\omega_1, \omega_2\}) = \omega_1^* + \omega_2^* + \omega_1^* \omega_2^* \\ &\neq \omega_1^* + \omega_2^* = P(A) + P(B). \end{aligned}$$

As usual, we call a function  $f : \Omega \rightarrow \mathbb{R}$  a *random variable*. A random variable corresponds to a measurement applied to the physical system described by  $(\Omega, \mathcal{A})$ . Denoting the Borel algebra of subsets of  $\mathbb{R}$  by  $\mathcal{B}(\mathbb{R})$ , for a random variable  $f$ , we define  $P^f : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{A}^*)$  by  $P^f(B) = P[f^{-1}(B)]$ . Thus,  $P^f = P \circ f^{-1}$  and we call  $P^f$  the *observable corresponding to  $f$* . The next result summarizes the properties of  $P(\cdot)$ .

**Theorem 4.4.** (a)  $P(A \cup B) = P(A) \vee P(B) = P(A) + P(B) + P(A)P(B)$  for all  $A, B \in \mathcal{A}$ . (b)  $P(\cdot)$  is unital, that is,  $P(\Omega) = I$ . (c)  $P(\cdot)$  is strongly monotone, that is,  $P(A) \leq P(B)$  if and only if  $A \subseteq B$ . (d)  $P(\cdot)$  is grade-2 additive, that is,

$$P(A \cup B \cup C) = P(A \cup B) + P(A \cup C) + P(B \cup C) + P(A) + P(B) + P(C).$$

**Proof.** (a) Letting  $A = \{\alpha_1, \dots, \alpha_r\}, B = \{\beta_1, \dots, \beta_s\}$  we have

$$\begin{aligned} P(A)P(B) &= [P(\alpha_1) + \dots + P(\alpha_r) + P(\alpha_1)P(\alpha_2) + \dots + P(\alpha_{r-1})P(\alpha_r)] \\ &\quad \cdot [P(\beta_1) + \dots + P(\beta_s) + P(\beta_1)P(\beta_2) + \dots + P(\beta_{s-1})P(\beta_s)] \\ &= \sum \{P(\alpha_i) : \alpha_i \in AB\} \\ &\quad + \sum \{P(\alpha_i)P(\beta_j) : \alpha_i \in AB', \beta_j \in A'B\}. \end{aligned}$$

It follows that

$$\begin{aligned} P(A \cup B) &= \sum \{P(\omega_i) : \omega_i \in A \cup B\} + \sum \{P(\omega_i)P(\omega_j) : \omega_i, \omega_j \in A \cup B, i < j\} \\ &= P(A) + P(B) + P(A)P(B). \end{aligned}$$

Applying theorem 4.1(c) we conclude that  $P(A \cup B) = P(A) \vee P(B)$ .

(b) Since

$$P(\Omega) = \sum_{i=1}^n P(\omega_i) + \sum_{i < j=1}^n P(\omega_i)P(\omega_j),$$

we see that  $P(\Omega)\omega_i^* = \omega_i$  for  $i = 1, \dots, n$  and  $P(\Omega)\omega_i^* \omega_j^* = \omega_i^* \omega_j^*$  for  $i, j = 1, \dots, n$ . Hence,  $P(\Omega)\phi = \phi$  for all  $\phi \in \mathcal{A}^*$  so  $P(\Omega) = I$ . (c) If  $A \subseteq B$ , then by (a) we have

$$P(B) = P(A \cup B) = P(A) + P(B) + P(A)P(B).$$

Hence,  $P(A)P(B) = P(B)P(A) = P(A)$  so  $P(A) \leq P(B)$ . Conversely, assume that  $P(A) \leq P(B)$  where  $A = \{\alpha_1, \dots, \alpha_r\}, B = \{\beta_1, \dots, \beta_s\}$ . Then

$$\begin{aligned} P(A) &= \alpha_1^* + \dots + \alpha_r^* + \alpha_1^* \alpha_2^* + \dots + \alpha_{r-1}^* \alpha_r^* = P(A)P(B) \\ &= (\alpha_1^* + \dots + \alpha_r^* + \alpha_1^* \alpha_2^* + \dots + \alpha_{r-1}^* \alpha_r^*)(\beta_1^* + \dots + \beta_s^* + \beta_1^* \beta_2^* + \dots + \beta_{s-1}^* \beta_s^*). \end{aligned}$$

If  $\alpha_i \notin B$  for some  $i = 1, \dots, r$ , then  $\alpha_i^*$  cannot appear in the product on the right-hand side which is a contradiction. Hence,  $\alpha_i \in B$  for  $i = 1, \dots, r$ , so  $A \subseteq B$ . (d) By (a) we have that

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B \cup C) + P(A)P(B \cup C) \\ &= P(A) + P(B) + P(C) + P(B)P(C) + P(A)[P(B) + P(C) + P(B)P(C)] \\ &= P(A) + P(B) + P(C) + P(B)P(C) + P(A)P(B) \\ &\quad + P(A)P(C) + P(A)P(B)P(C). \end{aligned}$$

Since  $A, B$  and  $C$  are mutually disjoint, we have that  $P(A)P(B)P(C) = 0$ . Hence, by (a) again, we conclude that

$$\begin{aligned} P(A \cup B) + P(A \cup C) + P(B \cup C) + P(A) + P(B) + P(C) \\ &= P(A) + P(B) + P(A)P(B) + P(A) + P(C) + P(A)P(C) \\ &\quad + P(B) + P(C) + P(B)P(C) + P(A) + P(B) + P(C) \\ &= P(A \cup B \cup C). \end{aligned}$$

Hence,  $P(\cdot)$  is grade-2 additive. □

**Example 1.** Letting  $\Omega = \{\omega_1, \omega_2\}$ , an ordered basis for  $\mathcal{A}^*$  is  $\omega_1^*, \omega_2^*, \omega_1^*\omega_2^*$ . In terms of this basis we have

$$P(\omega_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad P(\omega_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad P(\omega_1)P(\omega_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and  $P(\omega_1) + P(\omega_2) + P(\omega_1)P(\omega_2) = I$ . Projections need not commute. For instance, let  $Q$  be the projection:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then  $QP(\omega_2) = 0$  but

$$P(\omega_2)Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It follows from theorem 4.4 that if  $f : \Omega \rightarrow \mathbb{R}$  is a random variable, then the corresponding observable  $P^f : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{A}^*)$  is unital, strongly monotone, grade-2 additive and satisfies  $P^f(A \cup B) = P^f(A) \vee P^f(B)$  for all  $A, B \in \mathcal{B}(\mathbb{R})$ . Our observable terminology is at odds with the usual ‘intrinsic’ point of view which is observation (or measurement) independent and one can think of  $P(\cdot)$  as a mathematical construct and not refer to it as an observable.

### 5. Preclusion

For a physical system described by  $(\Omega, \mathcal{A})$ , there are frequently theoretical or experimental reasons for excluding certain sets  $A, B, \dots \in \mathcal{A}$  from consideration. Such sets are said to be *precluded*. For example, one may have an underlying quantum measure  $\mu$  on  $\mathcal{A}$  and sets of measure zero ( $\mu(A) = 0$ ) or sets of small measure ( $\mu(A) \approx 0$ ) may be precluded [5, 13–15]. In a physically realistic situation, precluded events should not occur. By convention we assume that  $\emptyset$  is precluded.

Let  $\mathcal{A}_p \subseteq \mathcal{A}$  be the set of precluded events. We say that a coevent  $\phi \in \mathcal{A}^*$  is *preclusive* if  $\phi(A) = 0$  for all  $A \in \mathcal{A}_p$ . The set of preclusive coevents form a subspace  $\mathcal{A}_p^*$  of  $\mathcal{A}^*$  and considering  $\mathcal{A}_p^*$  gives an important way of reducing the possible realities for a physical system [5, 14, 15]. We now present another way of reducing the possible realities. If  $\mathcal{A}_p = \{A_1, \dots, A_m\}$ , we say that  $\phi \in \mathcal{A}^*$  is *precluding* if  $P(A_1 \cup \dots \cup A_m)\phi = 0$ . Thus,  $\phi$  is precluding if and only if  $\phi$  is in the null space of  $P(A_1 \cup \dots \cup A_m)$ . This is again a subspace of  $\mathcal{A}^*$  which we will later show is contained in  $\mathcal{A}_p^*$ . Applying theorems 4.1(c) and 4.4(a) we have that

$$\begin{aligned} P(A_1 \cup \dots \cup A_m) &= \vee P(A_i) = [\wedge P(A_i)']' = [P(A_1)' \cdots P(A_m)']' \\ &= I + [I + P(A_1)] \cdots [I + P(A_m)]. \end{aligned}$$

It follows that  $\phi$  is precluding if and only if

$$P(A_1)' \cdots P(A_m)'\phi = \phi.$$

Thus, the precluding coevents are precisely the coevents in the range of the projection  $P(A_1)' \cdots P(A_m)'$ .

**Theorem 5.1.** (a) If  $P(A)\phi = 0$ , then  $\phi(A) = 0$  for  $A \in \mathcal{A}$ ,  $\phi \in \mathcal{A}^*$ .  
 (b) If  $\phi$  is precluding, then  $\phi$  is preclusive.

**Proof.** (a) Without loss of generality, we can assume that  $A = \{\omega_1, \dots, \omega_m\}$  and that

$$\phi = a_1\omega_1^* + \dots + a_n\omega_n^* + b_{12}\omega_1^*\omega_2^* + \dots + b_{n-1,n}\omega_{n-1}^*\omega_n^*$$

for  $a_i, b_{ij} \in \mathbb{Z}_2$ . Since

$$P(A)\phi = [P(\omega_1) + \dots + P(\omega_m) + P(\omega_1)P(\omega_2) + \dots + P(\omega_{m-1})P(\omega_m)]\phi = 0,$$

we have that  $a_1 = \dots = a_m = 0$ ,  $b_{ij} = 0$  for  $i, j \leq m$  and  $a_j + b_{ij} = 0$  for  $j > m, i \leq m$ . We conclude that  $\phi$  has the form

$$\begin{aligned} \phi &= a_{m+1}\omega_{m+1}^* + \dots + a_n\omega_n^* + a_{m+1}\omega_{m+1}^*\omega_1^* + \dots + a_{m+1}\omega_{m+1}^*\omega_m^* \\ &\quad + a_{m+2}\omega_{m+2}^*\omega_1^* + \dots + a_{m+2}\omega_{m+2}^*\omega_m^* + a_n\omega_n^*\omega_1^* + \dots + a_n\omega_n^*\omega_m^* \\ &\quad + b_{m+1,m+2}\omega_{m+1}^*\omega_{m+2}^* + \dots + b_{n-1,n}\omega_{n-1}^*\omega_n^*. \end{aligned}$$

It follows that  $\phi(A) = 0$ . (b) Assume that  $\mathcal{A}_p = \{A_1, \dots, A_m\}$ . If  $\phi$  is precluding, then  $P(A_1 \cup \dots \cup A_m)\phi = 0$ . By theorem 4.4(c),  $P(\cdot)$  is monotone so that

$$P(A_i)\phi = P(A_i)P(A_1 \cup \dots \cup A_m)\phi = 0$$

for  $i = 1, \dots, m$ . By (a) we have that  $\phi(A_i) = 0, i = 1, \dots, m$ . Hence,  $\phi \in \mathcal{A}_p^*$ . □

A *precluding basis* is a set  $S$  of precluding coevents such that every precluding coevent is a sum of elements of  $S$ . The definition of a preclusive basis is similar. Although such bases are not unique, they give an efficient way of describing all precluding (or preclusive) coevents.

**Example 2.** Let  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  and  $\mathcal{A}_p = \{\emptyset, \{\omega_1, \omega_2\}\}$ . It is easy to check that a preclusive basis consists of  $\omega_3^*, \omega_1^*\omega_3^*, \omega_2^*\omega_3^*, \omega_1^* + \omega_2^*, \omega_1^* + \omega_1^*\omega_2^*$ . To find the precluding coevents, we let  $A = \{\omega_1, \omega_2\}$  and solve the equation  $P(A)\phi = 0$ . Thus,

$$\begin{aligned} [P(\omega_1) + P(\omega_2) + P(\omega_1)P(\omega_2)](a\omega_1^* + b\omega_2^* + c\omega_3^* + d\omega_1^*\omega_2^* + e\omega_1^*\omega_3^* + f\omega_2^*\omega_3^*) \\ = a\omega_1^* + b\omega_1^*\omega_2^* + c\omega_1^*\omega_3^* + d\omega_1^*\omega_2^* + e\omega_1^*\omega_3^* + a\omega_1^*\omega_2^* + b\omega_2^* \\ + c\omega_2^*\omega_3^* + d\omega_1^*\omega_2^* + f\omega_2^*\omega_3^* + a\omega_1^*\omega_2^* + b\omega_1^*\omega_2^* + d\omega_1^*\omega_2^* \\ = 0. \end{aligned}$$

It follows that  $a = b = d = 0, c + e = c + f = 0$ . Hence,

$$\phi = c\omega_3^* + c\omega_1^*\omega_3^* + c\omega_2^*\omega_3^*$$

so the only nonzero precluding coevent is

$$\phi = \omega_3^* + \omega_1^*\omega_3^* + \omega_2^*\omega_3^*.$$

Of course,  $\phi$  is a precluding basis. Note that  $\phi$  is unital. This example shows that preclusive coevents need not be precluding.

**Example 3.** Let  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  and  $\mathcal{A}_p = \{\emptyset, \{\omega_1\}, \{\omega_2\}\}$ . A preclusive basis consists of  $\omega_3^*, \omega_1^*\omega_3^*, \omega_2^*\omega_3^*, \omega_1^*\omega_2^*$ . The only nonzero precluding coevent is  $\phi$  obtained in example 2. This is because

$$A = \{\omega_1, \omega_2\} = \{\omega_1\} \cup \{\omega_2\}.$$

**Example 4.** Let  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  and let  $\mathcal{A}_p = \{\emptyset, A\}$ , where  $A = \{\omega_1\}$ . To find the precluding coevents, we solve the equation  $P(A)\phi = 0$ . Thus,

$$\begin{aligned} P(\omega_1)[a\omega_1^* + b\omega_2^* + c\omega_3^* + d\omega_1^*\omega_2^* + e\omega_1^*\omega_3^* + f\omega_2^*\omega_3^*] \\ = a\omega_1^* + b\omega_1^*\omega_2^* + c\omega_1^*\omega_3^* + d\omega_1^*\omega_2^* + e\omega_1^*\omega_3^* = 0. \end{aligned}$$

Hence,  $a = b + d = c + e = 0$ . We conclude that  $\phi$  has the form

$$\phi = b\omega_2^* + c\omega_3^* + b\omega_1^*\omega_2^* + c\omega_1^*\omega_3^* + f\omega_2^*\omega_3^*.$$

Thus, a precluding basis consists of  $\omega_2^*\omega_3^*, \omega_2^* + \omega_1^*\omega_2^*$  and  $\omega_3^* + \omega_1^*\omega_3^*$ . The last two are not unital but sums with  $\omega_2^*\omega_3^*$  are unital.

We now discuss events  $B$  that can actually occur. That is there exists a preclusive or precluding coevent  $\phi$  such that  $\phi(B) = 1$ . It would be nice if whenever  $B$  is not precluded, then such a  $\phi$  exists. But this is asking too much as simple examples show. However, we do have the following result which gives a kind of duality between preclusive and precluding coevents.

**Theorem 5.2.** Let  $\mathcal{A}_p = \{A_1, \dots, A_m\}$ ,  $A = A_1 \cup \dots \cup A_m$  and  $B \in \mathcal{A}$ .

- (a) If  $BA' \neq \emptyset$ , then there exists a preclusive coevent  $\phi$  such that  $\phi(B) = 1$ .
- (b) If there exists a precluding coevent  $\phi$  such that  $\phi(B) = 1$ , then  $BA' \neq \emptyset$ .

**Proof.** (a) If  $\omega \in BA'$ , then  $\omega \in BA'_i, i = 1, \dots, m$ . Hence,  $\omega^*(B) = 1$  and  $\omega^*(A_i) = 0, i = 1, \dots, m$ . We conclude that  $\omega^*$  is preclusive. (b) Suppose  $BA' = \emptyset$ . Then  $B \subseteq A$  and if  $\phi$  is precluding, then  $P(A)\phi = 0$ . Hence, by theorem 4.4(c) we have

$$P(B)\phi = P(B)P(A)\phi = 0.$$

Applying theorem 5.1(a), we conclude that  $\phi(B) = 0$ . Hence, there is no precluding coevent  $\phi$  such that  $\phi(B) = 1$ . We have thus proved the contrapositive of (b) so (b) holds.  $\square$

**Corollary 5.3.** Let  $\mathcal{A}_p = \{A_1, \dots, A_m\}$ ,  $A = A_1 \cup \dots \cup A_m$  and  $B \in \mathcal{A}$ . (a) If  $\phi(B) = 0$  for every preclusive  $\phi$ , then  $B \subseteq A$ . (b) If  $B \subseteq A$ , then  $\phi(B) = 0$  for every precluding  $\phi$ .

The result in theorem 5.2(a) does not hold if preclusive is replaced by precluding. In example 4,  $A_1 = \{\omega_1\}$  is the only nonempty precluded event. Letting  $B = \{\omega_1, \omega_2\}$  we have that  $BA' = \{\omega_2\} \neq \emptyset$ . However, all the precluding coevents listed in example 4 vanish on  $B$ . Hence,  $\phi(B) = 0$  for all precluding coevents. The result in theorem 5.2(b) does not hold if precluding is replaced by preclusive. In example 2, letting  $A = \{\omega_1, \omega_2\}$  and  $B = \{\omega_1\}$ ,  $\phi = \omega_1^* + \omega_2^*$  is preclusive and  $\phi(B) = 1$ . However,  $BA' = \emptyset$ .



Examples 2 and 3 have the pleasant feature that there is a unique nonzero precluding coevent. However, the next example shows that there can be many preclusive coevents and no nonzero precluding coevent.

**Example 5.** In the three-slit experiment  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  considered previously, suppose  $\{\omega_1, \omega_2\}$  and  $\{\omega_2, \omega_3\}$  are the only nonempty precluded events. Since

$$\Omega = \{\omega_1, \omega_2\} \cup \{\omega_2, \omega_3\},$$

we have that  $\phi$  is precluding if and only if  $\phi = P(\Omega)\phi = 0$  so the only precluding coevent is 0. However, there are many preclusive coevents. For example,  $\omega_1^* + \omega_2^* + \omega_3^*$ ,  $\omega_1^* + \omega_1^*\omega_2^*$ ,  $\omega_3^* + \omega_2^*\omega_3^*$ ,  $\omega_1^*\omega_3^*$  form a preclusive basis.

It should be mentioned that in previous studies it has usually been assumed that the union of mutually disjoint precluded events is precluded. However, we did not make this assumption here.

### Acknowledgment

The author thanks R Sorkin for reading a preliminary version of this manuscript, correcting various errors and suggesting improvements.

### References

- [1] Birkhoff G and von Neumann J 1936 The logic of quantum mechanics *Ann. Math.* **37** 823–34
- [2] Drapal A and Vojtěchovský P 2009 Symmetric multilinear forms and polarization of polynomials *Linear Algebra Appl.* **431** 998–1012
- [3] Dvurečenskij A and Pulmannová S 2000 *New Trends in Quantum Structures* (Dordrecht: Kluwer)
- [4] Gell-Mann M and Hartle J B 1993 Classical equations for quantum systems *Phys. Rev. D* **47** 3345–82
- [5] Ghazi-Tabatabai Y 2009 Quantum measure theory: a new interpretation arXiv:quant-ph/0906.0294
- [6] Griffiths R B 1984 Consistent histories and the interpretation of quantum mechanics *J. Stat. Phys.* **36** 219–72
- [7] Gudder S 1988 *Quantum Probability* (New York: Academic)
- [8] Gudder S 2010 Finite quantum measure spaces *Am. Math. Mon.* submitted
- [9] Jauch J M 1968 *Foundations of Quantum Mechanics* (Reading, MA: Addison-Wesley)
- [10] Mackey G W 1963 *Mathematical Foundations of Quantum Mechanics* (New York: Benjamin)
- [11] Piron C 1968 *Foundations of Quantum Physics* (Reading, MA: Addison-Wesley)
- [12] Salgado R 2002 Some identities for the quantum measure and its generalizations *Mod. Phys. Lett. A* **17** 711–28
- [13] Sorkin R 1994 Quantum mechanics as quantum measure theory *Mod. Phys. Lett. A* **9** 3119–27
- [14] Sorkin R 2007 Quantum mechanics without the wave function *J. Phys. A: Math. Theor.* **40** 3207–31
- [15] Sorkin R An exercise in ‘anhomomorphic logic’ *J. Phys.: Condens. Matter* arXiv:quant-ph/0703276v2 submitted
- [16] Surya S and Wallden P 2008 Quantum covers in quantum measure theory arXiv:quant-ph/0809.1951
- [17] Ududec C, Barnum H and Emerson J 2009 Three slit experiments and the structure of quantum theory arXiv:quant-ph/0909.4787
- [18] Varadarajan V S 1968 *Geometry of Quantum Theory* (Princeton, NJ: van Nostrand)